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is known as the Taylor's remainder R_n after n terms and is due Schlomilch and Roche.

(ii) Putting p = 1. we obtain

$$R_n = \frac{h^{n-1} (1-\theta)^{n-1}}{(n-1)!} f^n (a+\theta h),$$

which form of remainder is due to Cauchy.

(iii) Putting p = n, we obtain

$$R_n = \frac{h^n}{n!} f^n (a + \theta h),$$

which is due to Lagrange.

Cor. 1. Let x be a point of the interval [a, a + h]. Let f satisfy the conditions of Taylor's theorem in [a, a + h] so that it satisfies the conditions for [a, x] also.

Changing a + h to x i.e., h to x - a, in (i), we obtain

$$f(x) = f(a) + (x - a)f'(a) + \frac{(x - a)^2}{2!}f''(a) + \frac{(x - a)^3}{3!}f'''(a) + \dots + \frac{(x - a)^{n-1}}{(n-1)!}f^{n-1}(a) + \frac{(x - a)^n(1 - \theta)^{n-p}}{p \cdot (n-1)!}f''[a + \theta(x - a)], 0 < \theta < 1.$$

This result holds $\forall x \in [a, a+h]$. Of course, θ may be differe for different points x.

Cor. 2. Maclaurin's theorem. Putting a=0, we see that $x \in [0, h]$, then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{n-1}(0) + \frac{x^n(1-\theta)^{n-p}}{p(n-1)!}f''(\theta)$$

which holds when

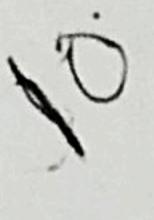
(i) f^{n-1} is Continuous in [0, h], and (ii) f'' exists in]0, h[.

Putting p = 1 and p = n, respectively in the Schlomilch form remainder

$$\frac{x^{n}(1-\theta)^{n-p}f^{n}(\theta x)}{p.(n-1)!}$$

we see that Cauchy's and Lagrange's forms are respectively

$$\frac{x^{n}(1-\theta)^{n-1}}{(n-1)!}f^{n}(\theta x) \text{ and } \frac{x^{n}}{n!}f^{n}(\theta x).$$



(i) the (n-1)th derivative f^{n-1} is continuous in [a, a+h],

(ii) the nth derivative f^n exists in]a, a + h[.

and (iii) p is a given positive integer,

and (iii) p is a given positive integer,
then there exists at least one number,
$$\theta$$
, between 0 and 1 such that
$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$

$$+ \frac{h^2(1-\theta)^{n-p}}{(n-1)!} f^n(a+\theta h). \qquad \dots (i)$$

The condition (i) implies the continuity of

$$f, f', f'', \dots, f^{n-2}$$
 in $[a, a+h]$.

Let a function ϕ be defined by

$$\varphi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \dots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

where A is a constant to be determined such that

$$\varphi(a) = \varphi(a+h).$$

Thus A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^p$$
...(ii)

The function φ is continuous in [a, a + h], derivable in]a, a + h[and $\varphi(a) = \varphi(a + h)$. Hence, by Rolle's theorem, there exists at least one number, θ , between 0 and 1 such that

$$\Phi'(a + \theta h) = 0$$
But
$$\Phi'(x) = \frac{(a + h - x)^{n-1}}{(n-1)!} f^{n}(x) - pA (a + h - x)^{p-1}.$$

$$\therefore 0 = \varphi'(a + \theta h) = \frac{h^{-1}(1 - \theta)^{n-1}}{(n-1)!} f^n(a + \theta h) - pA(1 - \theta)^{p-1} h^{p-1}$$

$$\Rightarrow A = \frac{h^{n-p} (1-\theta)^{n-p}}{p \cdot (n-1)!} \cdot f^n (a+\theta h). \text{ for } (1-\theta) \neq 0 \text{ and } h \neq 0.$$

Substituting the value of .A in (ii), we get the required result (i).

(i) Remainder after r terms. The term

$$R_n = \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^n (a+\theta h),$$