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Taylor and Binomial Series.

B Sc. Part II

is known as the Taylor's remainder R_n after n terms and is due to Schlomilch and Roche.

(ii) Putting $p = 1$. we obtain

$$R_n = \frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)\,!} f^n(a+\theta h),$$

which form of remainder is due to Cauchy.

(iii) Putting $p = n$, we obtain

$$R_n = \frac{h^n}{n\,!} f^n(a+\theta h),$$

which is due to Lagrange.

Cor. 1. Let x be a point of the interval $[a, a+h]$. Let f satisfy the conditions of Taylor's theorem in $[a, a+h]$ so that it satisfies the conditions for $[a, x]$ also.

Changing $a + h$ to x i.e., h to $x - a$, in (i), we obtain

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{2\,!} f''(a) + \frac{(x-a)^3}{3\,!} f'''(a) + \ldots +$$
$$\frac{(x-a)^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{(x-a)^n (1-\theta)^{n-p}}{p\,.\,(n-1)!} f^n[a+\theta(x-a)],\ 0<\theta<1.$$

This result holds $\forall\ x \in [a, a+h]$. Of course, θ may be differe[nt] for different points x.

Cor. 2. Maclaurin's theorem. Putting $a = 0$, we see that [if] $x \in [0, h]$, then

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3\,!} f'''(0) + \ldots +$$
$$\frac{x^{n-1}}{(n-1)\,!} f^{n-1}(0) + \frac{x^n (1-\theta)^{n-p}}{p\,(n-1)\,!} f^n(\theta$$

which holds when

(i) f^{n-1} is Continuous in $[0, h]$, and (ii) f^n exists in $]0, h[$.

Putting $p = 1$ and $p = n$, respectively in the Schlomilch form [of] remainder

$$\frac{x^n (1-\theta)^{n-p} f^n(\theta x)}{p\,.\,(n-1)\,!}$$

we see that Cauchy's and Lagrange's forms are respectively

$$\frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^n(\theta x) \text{ and } \frac{x^n}{n\,!} f^n(\theta x).$$

(i) the $(n-1)$th derivative f^{n-1} is continuous in $[a, a+h]$,

(ii) the nth derivative f^{n} exists in $]a, a+h[$.

and (iii) p is a given positive integer,

then there exists at least one number, θ, between 0 and 1 such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a)$$
$$+ \frac{h^2(1-\theta)^{n-p}}{(n-1)!\,p} f^{n}(a+\theta h). \qquad \ldots\text{(i)}$$

The condition (i) implies the continuity of

$f, f', f'', \ldots\ldots, f^{n-2}$ in $[a, a+h]$.

Let a function φ be defined by

$$\varphi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots$$
$$+ \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n-1}(x) + A(a+h-x)^p$$

where A is a constant to be determined such that

$$\varphi(a) = \varphi(a+h).$$

Thus A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(a) + Ah^p \qquad \ldots\text{(ii)}$$

The function φ is continuous in $[a, a+h]$, derivable in $]a, a+h[$ and $\varphi(a) = \varphi(a+h)$. Hence, by Rolle's theorem, there exists at least one number, θ, between 0 and 1 such that

$$\varphi'(a+\theta h) = 0$$

But $$\varphi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{n}(x) - pA(a+h-x)^{p-1}.$$

$$\therefore\ 0 = \varphi'(a+\theta h) = \frac{h^{-1}(1-\theta)^{n-1}}{(n-1)!}f^{n}(a+\theta h) - pA(1-\theta)^{p-1}h^{p-1}$$

$$\Rightarrow A = \frac{h^{n-p}(1-\theta)^{n-p}}{p\,.\,(n-1)!}\,.f^{n}(a+\theta h).\ \text{for } (1-\theta) \neq 0 \text{ and } h \neq 0.$$

Substituting the value of A in (ii), we get the required result (i).

(i) Remainder after r terms. The term

$$R_n = \frac{h^n(1-\theta)^{n-p}}{p\,(n-1)!}f^{n}(a+\theta h),$$